find a set of locations and the corresponding voltages that give us the best correction to the surface thermal distortions of the primary mirror under a given type of thermal loads; the other is to find one set of locations and corresponding voltages that provide the best correction to the surface thermal distortions caused by all of the four different kinds of thermal loads. The two types of problems are difficult and computationally intensive. The second type is a more challenging, multicriterion optimization problem. The search space for these problems is highly multimodal, and conventional point-by-point optimization techniques usually get stuck at the local optimum, but population-based GAs are very good at searching such space and more likely get better results than the traditional techniques. The search space of 1.38×10^{54} different sets of actuator locations for the case of 121 piezoelectric actuators is much larger than that for the case of 30 piezoelectric actuators, but the GA version 3 still converged very fast and finally found a very good set of actuator locations that can be used to reduce all of the four kinds of thermal distortions. The results show that the two modifications employed in this study significantly improve the performance of GAs. The current version of GAs are more effective than the preceding versions in solving optimization problems of determining actuator locations for thermal distortion control.

The problems for this study are computationally intensive. Getting one solution using the GAs with the limit of 15,000 evaluations for the second optimization problem took more than a week. GAs are very general and robust optimization methods that can be applied to virtually any optimization problem. Parallel GAs can significantly reduce the time and be more natural to imitate the evolution of the nature, and so our current research is to develop parallel GAs.

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A. Chattopadhyay
Associate Editor

Stability of Tapered Columns Under End-Concentrated and Variably Distributed Follower Forces

H. Qiao* and Q. S. Li†
City University of Hong Kong, Hong Kong, People's Republic of China

Introduction

ELASTIC systems subjected to concentrated and distributed types of nonconservative follower forces are always encountered in engineering practices, such as in stability analysis of rockets, missiles, slender space structures, pipes conveying fluid, automobile disk, and drum brakes. A literature review indicates that the nonconservative stability of nonuniform structures was usually solved by the Galerkin method,[1] the finite difference method,[2] the finite element method (FEM),[3] and the transfer matrix method.[4] It is difficult to find the exact analytical solutions for stability of tapered structures with arbitrary boundary conditions. In this Note a successful attempt is made to present an efficient analytical method for the nonconservative stability of tapered columns. The closed-form solutions for the stability of two types of nonuniform columns subjected to an end concentrated and variably distributed follower forces are derived for the first time. The advantage of the proposed method is that the resulting characteristic equation for stability of a nonuniform column with any kind of two-end support configuration can be conveniently determined from a second-order determinant. As a consequence, the decrease in the determinant order, as compared with previously developed procedures, leads to significant savings in the computational effort.

Theory

A column with variable cross section under the combined action of an end concentrated follower force and variably distributed follower forces along the column is shown in Fig. 1. Considering the element shown in Fig. 2 and according to the d'Alembert principle, all of the forces acting on the element should satisfy the equilibrium conditions. From $\sum F_x = 0$, $\sum F_y = 0$, and using the method of separation of variables, one obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[K(x)\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2}\right] + N(x)\frac{\mathrm{d}^2X(x)}{\mathrm{d}x^2} - \bar{m}(x)\omega^2X(x) = 0 \tag{1}$$

where $X(x)$ is the mode shape function, ω is the circular natural frequency, $\bar{m}(x)$ is the mass per unit length of the column, $K(x)$ is the flexural stiffness, and $N(x)$ is the axial force.

Obviously, the solution of Eq. (1) is dependent on the expression of $K(x)$, $N(x)$, and $\bar{m}(x)$. As suggested by Li et al.[5,6] and Li,[7] the functions for describing the variations of $K(x)$, $N(x)$, and $\bar{m}(x)$, for many cases of structural members are power functions and exponential functions, which are considered in this Note.

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*Assistant Professor, Department of Manufacturing Engineering and Engineering Management, Tat Chee Avenue, Kowloon.

†Associate Professor, Department of Building and Construction, Tat Chee Avenue, Kowloon.

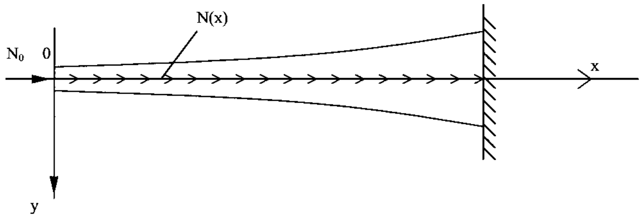

Fig. 1 Nonuniform column with C-F ends.

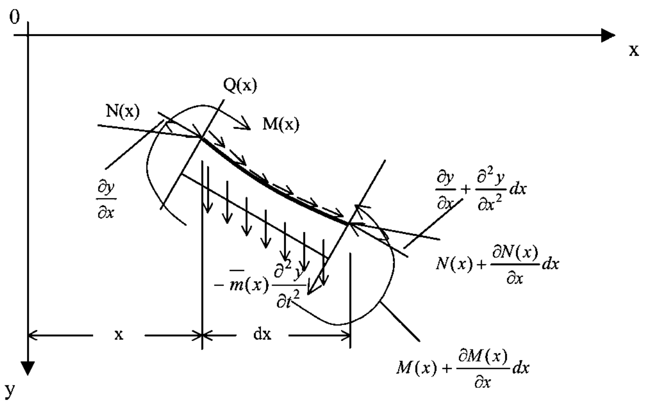

Fig. 2 Element.

Case 1

The functions for describing the distributions of the flexural stiffness, axial force, and mass intensity are expressed as

$$K(x) = K_0[1+\beta(x/L)]^{n+4}, \qquad N(x) = N_0[1+\beta(x/L)]^{n+2}$$
$$\bar{m}(x) = \bar{m}_0[1+\beta(x/L)]^{n} \tag{2}$$

where K_0, N_0, and $\bar{m}_0$ denote the flexural stiffness, axial force, and mass per unit length of the column at $x = 0$, respectively; L is the length of the column.

Substituting Eq. (2) into Eq. (1) and setting $\xi = \ln[1+\beta(x/L)]$ results in

$$\frac{d^4X(\xi)}{d\xi^4} + 2(n+4)\frac{d^3X(\xi)}{d\xi^3} + \left[(n+4)(n+3) + \frac{N_0L^2}{K_0\beta^2}\right]$$
$$\times \frac{d^2X(\xi)}{d\xi^2} - \frac{\bar{m}_0\omega^2L^4}{K_0\beta^4}X(\xi) = 0 \tag{3}$$

The general solution of Eq. (3) can be written as

$$X(\xi) = \sum_{i=1}^{4} C_i \exp(r_i\xi) \tag{4}$$

where

$$r_{1,2,3,4} = -\tfrac{1}{2}\left(n+4\pm\sqrt{f}\right)$$
$$\pm\sqrt{\tfrac{1}{4}\left(n+4\pm\sqrt{f}\right)^2 - \tfrac{1}{2}\left(y_1\pm\sqrt{y_1^2-4e_1}\right)}$$
$$e_1 = -\alpha_1^2/\beta^4, \qquad \alpha_1^2 = \bar{m}_0\omega^4L^4/K_0$$
$$f = n+4+y_1-\alpha_2/\beta^2, \qquad p = 4\alpha_1^2/\beta^4 - c^2/3$$
$$y_1 = \sqrt[3]{-q/2+\sqrt{(q/2)^2+(p/3)^3}}$$
$$+\sqrt[3]{-q/2-\sqrt{(q/2)^2+(p/3)^3}}$$
$$q = \left(\alpha_1^2/\beta^4\right)\left[4\left(n+4-\alpha_2/\beta^2\right)+4c/3\right] - 2c^3/27$$
$$c = (n+4)(n+3)+\alpha_2/\beta^2, \qquad \alpha_2^2 = N_0L^2/K_0$$

The sign before $\sqrt{f}$ is the same as that before $\sqrt{(y_1^2-4e_1)}$ if $(n+4)y_1 > c$; otherwise, the signs should be different.

Case 2

The distributions of flexural stiffness, axial force, and mass intensity are given by

$$K(x) = K_0\exp[b(x/L)], \qquad N(x) = N_0\exp[b(x/L)]$$
$$\bar{m}(x) = \bar{m}_0\exp[b(x/L)] \tag{5}$$

Substituting Eq. (5) into Eq. (1) leads to

$$\frac{d^4X(x)}{dx^4} + 2\frac{b}{L}\frac{d^3X(x)}{dx^3}$$
$$+\left[\left(\frac{b}{L}\right)^2 + \frac{N_0}{K_0}\right]\frac{d^2X(x)}{dx^2} - \frac{\bar{m}_0\omega^2}{K_0}X(x) = 0 \tag{6}$$

The general solutions of Eq. (6) can also be expressed in the form of Eq. (4), but for this case we have

$$r_{1,2,3,4} = -\tfrac{1}{2}\left(b/L\pm\sqrt{y_1-N_0/K_0}\right)$$
$$\pm\sqrt{\tfrac{1}{16}\left(b/L\pm\sqrt{y_1-N_0/K_0}\right)^2 - \tfrac{1}{2}\left(y_1\pm\sqrt{y_1^2-4e}\right)} \tag{7}$$

where $e = -\bar{m}_0\omega^2/K_0$ and y_1 is already defined; the other parameters in Eq. (7) are

$$q = 4e(d/3 - N_0/K_0) - 2d^3/27, \qquad p = 4e - d^2/3$$
$$d = (b/L)^2 + N_0/K_0$$

If $2b/L > d$, the sign before $\sqrt{(y_1 - N_0/K_0)}$ is the same as that before $\sqrt{(y_1^2-4e)}$; otherwise, the signs should be different.

Substituting $b = 0$ into Eqs. (5) and (6) results in a special case that represents a uniform cantilever column subjected to an end concentrated follower force. The general solutions for this special case are found as

$$X(x) = C_1\sin k_1x + C_2\cos k_1x + C_3\sinh k_2x + C_4\cosh k_2x \tag{8}$$

where

$$k_1 = \lambda_1\sqrt{\sqrt{1+(k/\lambda_1)^4}+1}, \qquad k_2 = \lambda_1\sqrt{\sqrt{1+(k/\lambda_1)^4}-1}$$
$$\lambda_1^2 = N_0/2K_0, \qquad k^4 = \bar{m}_0\omega^2/K_0$$

$S_1(x)(i = 1\sim4)$ are used here to denote the linearly independent solutions of Eqs. (3) and (6); then the general solution for cases 1 and 2 can be expressed as

$$X(x) = \sum_{i=1}^{4} C_iS_i(x) \tag{9}$$

where C_i are four integration constants to be determined from the boundary conditions.

To simplify the analysis, based on the derived solutions just presented, four linearly independent fundamental solutions $\bar{S}_i(x)$ are chosen such that they satisfy the following normalization conditions at the origin of the co-ordinate system:

$$\begin{bmatrix} \bar{S}_1(0) & \bar{S}_1'(0) & \bar{S}_1''(0) & \bar{S}_1'''(0) \\ \bar{S}_2(0) & \bar{S}_2'(0) & \bar{S}_2''(0) & \bar{S}_2'''(0) \\ \bar{S}_3(0) & \bar{S}_3'(0) & \bar{S}_3''(0) & \bar{S}_3'''(0) \\ \bar{S}_4(0) & \bar{S}_4'(0) & \bar{S}_4''(0) & \bar{S}_4'''(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

where the primes indicate differentiation with respect to the coordinate variable x. $\bar{S}_i(x)(i = 1\sim4)$ can be easily constructed by

$$\begin{bmatrix} \bar{S}_1(x) \\ \bar{S}_2(x) \\ \bar{S}_3(x) \\ \bar{S}_4(x) \end{bmatrix} = \begin{bmatrix} S_1(0) & S_1'(0) & S_1''(0) & S_1'''(0) \\ S_2(0) & S_2'(0) & S_2''(0) & S_2'''(0) \\ S_3(0) & S_3'(0) & S_3''(0) & S_3'''(0) \\ S_4(0) & S_4'(0) & S_4''(0) & S_4'''(0) \end{bmatrix}^{-1} \begin{bmatrix} S_1(x) \\ S_2(x) \\ S_3(x) \\ S_4(x) \end{bmatrix} \tag{11}$$

The advantage of using the fundamental solutions $\bar{S}_i(x)$ is that the mode shape functions can be easily expressed in terms of the initial parameters and $\bar{S}_i(x)$ as follows:

$$X(x) = X(0)\bar{S}_1(x) + \varphi(0)\bar{S}_2(x) - [M(0)/K(0)]\bar{S}_3(x) - [1/K(0)][Q(0) - \mu(0)M(0)]\bar{S}_4(x) \tag{12}$$

where $\mu(x) = K'(x)/K(x)$, $X(0)$, $\varphi(0)$, $M(0)$, and $Q(0)$ are the initial displacement, slope, bending moment, and shear force at $x = 0$, respectively.

Because two of the four initial parameters are known for any type of support condition, it is easy to establish the characteristic equation for the nonconservative stability of a nonuniform column with classical or nonclassical boundary conditions as follows:

1) A nonuniform column with hinged-hinged end conditions. The boundary conditions for this case are given by

$$X(0) = 0, \qquad M(0) = 0 \tag{13}$$

$$X(L) = 0, \qquad M(L) = 0 \quad \text{or} \quad X''(L) = 0 \tag{14}$$

Applying the boundary condition at $x = 0$ to Eq. (12) results in

$$X(x) = \varphi(0)\bar{S}_2(x) - [Q(0)/K(0)]\bar{S}_4(x) \tag{15}$$

Using the boundary conditions at $x = L$ and Eq. (15), one obtains the characteristic equation

$$\bar{S}_2(L)\bar{S}_4''(L) - \bar{S}_2''(L)\bar{S}_4(L) = 0 \tag{16}$$

2) A nonuniform column with clamped-free (C-F) ends (shown in Fig. 1) and a concentrated mass m_0 attached at the free end. The boundary conditions for this case are given by

$$M(0) = 0, \qquad Q(0) = -m_0\omega^2 X(0) \tag{17}$$

$$X(L) = 0, \qquad \varphi(L) = 0 \tag{18}$$

Applying the boundary conditions at $x = 0$ to Eq. (12) leads to

$$X(x) = X(0)\left\{\bar{S}_1(x) + \left[m_0\omega^2/K(0)\right]\bar{S}_4(x)\right\} + \varphi(0)\bar{S}_2(x) \tag{19}$$

The characteristic equation can be established using Eqs. (18) and (19) as follows:

$$\bar{S}_2'(L)\left\{\bar{S}_1(L) + \left[m_0\omega^2/K(0)\right]\bar{S}_4(L)\right\} - \bar{S}_2(L)\left\{\bar{S}_1'(L) + \left[m_0\omega^2/K(0)\right]\bar{S}_4'(L)\right\} = 0 \tag{20}$$

Setting $m_0 = 0$ results in the characteristic equation of a nonuniform column with C-F ends and without the concentrated mass.

Numerical Example

A nonuniform cantilever column subjected to an end-concentrated follower force N_0 and variably distributed follower forces $N(x)$ along the column is shown in Fig. 1. The variations in flexural stiffness, axial force, and mass are described by Eq. (2). The four linearly independent solutions for this example are given in Eq. (4), and the fundamental solutions can be constructed by Eq. (11); the characteristic equation for this case is Eq. (16).

The critical buckling force can be written as $N_{0,\mathrm{cr}} = \alpha(\beta)\,[K(L)/L^2]$, where $\alpha(\beta)$ is a coefficient depending on the taper ratio of a column if the boundary conditions and the distributions of mass, stiffness, and axial forces are given.

When $\beta = 0.2, 0.5$, it is determined that $\alpha(\beta) = 9.9112, 3.6673$, respectively, illustrating the effect of taper ratio of the nonuniform column on its critical buckling force is significant. The FEM with cubic approximation of displacements is also adopted to conduct the stability analysis for comparison purposes. The column is divided into 40 uniform elements for the stability analysis. It is found that when $\beta = 0.2, 0.5$, $\alpha(\beta) = 9.9111, 3.6672$, respectively. Obviously, the results obtained from the proposed method and FEM are in close agreement, but it is observed that the proposed method takes less computational time than FEM, illustrating that the present method is efficient, convenient, and accurate.

The effect of the taper ratio on the critical buckling force is shown in Fig. 3. It is evident that the critical buckling force decreases as the taper ratio of the column increases.

If there is a concentrated mass m_0 attached at the free end, the characteristic equation for this case is given in Eq. (20). The influence of the end-concentrated mass on the critical buckling force is shown in Fig. 4. It is clear that the critical buckling force decreases as the ratio of m_0 to M increases, where M is the total mass of the column.

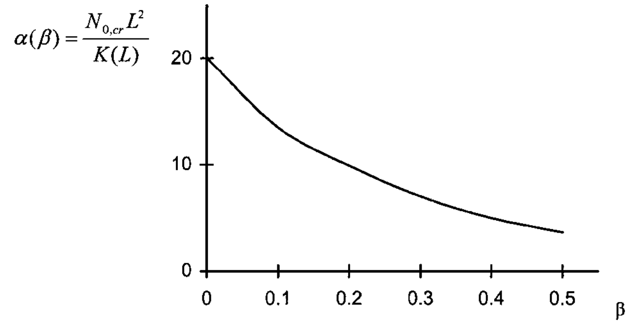

Fig. 3 Effect of the taper ratio on the critical buckling force.

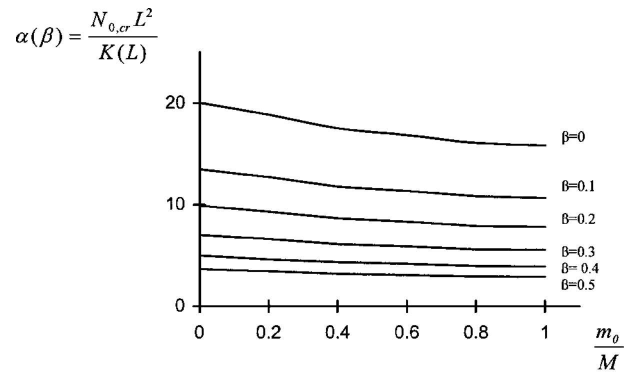

Fig. 4 Effect of the end-concentrated mass on the critical buckling force.

Conclusions

The closed-form stability solutions for two types of nonuniform columns and variably distributed followers are derived for the first time. The advantage of the proposed method is that the characteristic equation expressed in terms of the fundamental solutions for stability of a nonuniform column with arbitrary boundary conditions can be conveniently determined from a second-order determinant. As a result, the decrease in the determinant order leads to significant savings in the computational effort. The numerical example shows that the results from the proposed method are in good agreement with those from FEM, but the proposed method takes less computational time than FEM, illustrating that the present procedure is an exact and efficient method.

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A. N. Palazotto
Associate Editor